States and the Numerical Range in the Regular Algebra

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In this dissertation we investigate the algebra numerical range defined by the Banach algebra of regular operators on a Dedekind complete complex Banach lattice, i.e., $V(\mathcal{L}_r(E), T) = \{\Phi(T) : \Phi \in \mathcal{L}_r(E)^*, ||\Phi|| = 1 = \Phi(I)\}$. For T in the center $\mathcal{Z}(E)$ of E we prove that $V(\mathcal{L}_r(E), T) = \text{co}(\sigma(T))$. For $T \perp I$ we prove that $V(\mathcal{L}_r(E), T)$ is a disk centered at the origin. We then consider the part of $V(\mathcal{L}_r(E), T)$ obtained by restricting ourselves to positive states $\Phi \in \mathcal{L}_r(E)^*$. In this case we show that we get a closed interval on the real line.
Next we consider the problem of characterizing the linear maps on $\mathcal{L}_r(E)$ which preserve $V(\mathcal{L}_r(E), T)$. For this we first describe the regular states on $\mathcal{L}_r(E)$, in particular for the case $E = \ell_p(n)$ for $1 \leq p \leq \infty$. This description allows us to show that any map Ψ on $\mathcal{L}_r(\ell_p(n))$ preserving $V(\mathcal{L}_r(\ell_p(n)), T)$ for all $T \in \mathcal{L}_r(\ell_p(n))$ is of the form $\Psi(T) = U * (P^t \mathcal{Q} T P)$ where U consists of elements of modulus 1, $(*)$ represents Hadamard multiplication, P is a permutation, and $\mathcal{Q}$ is a map that permutes off-diagonal entries of T. Furthermore, special conditions are given for $\mathcal{Q}$ for the cases $p = 1$, $p = \infty$ and $p = 2$.
Finally, some extensions of these results to more general finite dimensional Banach lattices and infinite dimensional ℓ_p's are considered.